A POINT SOURCE AND A VORTEX FILAMENT IN A HELICAL CURRENT

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In the book of Vasil'ev [1] the problem is posed of investigating the two-dimensional or two-parameter helical motion, which depends on the two cylindrical coordinates r, z as determined by the equation for the stream function

$$\frac{\partial^2\psi}{\partial z^2} + r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\psi}{\partial r}\right) + k^2\psi = -kC \tag{1}$$

by means of which the components of the velocity are expressed thus:

$$v_r = \frac{1}{r}\frac{\partial\psi}{\partial z}, \qquad v_z = -\frac{1}{r}\frac{\partial\psi}{\partial z}, \qquad v_\varphi = \frac{k\psi + C}{r} \tag{2}$$

The motion is considered in the region $0 < z < \infty$ and $0 < r < \infty$ with boundary conditions

$$\psi(z, 0) = 0, \qquad \psi(0, r) = \psi_0 = \text{const} \tag{3}$$

Vasil'ev obtains the solution in two forms. The first form is

$$\psi = r\psi_0 \operatorname{Re}\int_0^\infty J_1(\lambda r)\exp\left(-z\sqrt{\lambda^2 - k^2}\right)d\lambda -$$

$$-kCr\operatorname{Re}\left\{\int_0^\infty \frac{J_1(\lambda r)}{k^2 - \lambda^2}\left[1 - \exp\left(-z\sqrt{\lambda^2 - k^2}\right)\right]d\lambda\right\} \tag{4}$$

Here for $\lambda < k$, $\sqrt{\lambda^2 - k^2}$ is taken as $i\sqrt{k^2 - \lambda^2}$ and consequently

$$\operatorname{Re}\left(e^{-z\sqrt{\lambda^2 - k^2}}\right) = \cos z\sqrt{k^2 - \lambda^2}$$

It is stated that the integral (4) can take a simple form, if formula (22) on p. 35 of the book [2] is used. This formula for $\nu = 0$ can be written in the form

$$A(r, z, k) = -\operatorname{Re}\int_0^\infty J_1(\lambda r)\frac{\exp(-z\sqrt{\lambda^2-k^2})}{\sqrt{\lambda^2-k^2}}d\lambda = \frac{1}{kr}(\sin kz - \sin k\sqrt{z^2+r^2}) \qquad (5)$$

taking the root $\sqrt{\lambda^2 - k^2}$ equal to $i\sqrt{k^2 - \lambda^2}$ for $\lambda < k$. It is possible also to assume

$$A(r, z, k) = \int_0^\infty J_1(\lambda r) f(\lambda, z, k)\, d\lambda$$

where

$$f(\lambda, z, k) = \begin{cases} \dfrac{\sin z\sqrt{k^2-\lambda^2}}{\sqrt{k^2-\lambda^2}} & (0<\lambda<k) \\ \dfrac{\exp(-z\sqrt{\lambda^2-k^2})}{\sqrt{\lambda^2-k^2}} & (k<\lambda<\infty) \end{cases} \qquad (6)$$

Now ψ is easily expressed by means of $A(r, z, k)$:

$$\psi = r\psi_0\frac{\partial A}{\partial z} - kCr\int_0^z A(r, \zeta, k)\, d\zeta \qquad (7)$$

and, as a result it is easy to express it in the form

$$\psi = \psi_0\left(\cos kz - \frac{z\cos k\sqrt{z^2+r^2}}{\sqrt{z^2+r^2}}\right) - \frac{C}{k}\left(1 - \cos kz - k\int_0^z \sin k\sqrt{\zeta^2+r^2}\, d\zeta\right) \qquad (8)$$

It is easily shown that conditions (3) are satisfied, so that ψ is a bounded function. For $k = 0$ we have

$$\psi = \psi_0\left(1 - \frac{z}{\sqrt{z^2+r^2}}\right) \qquad (9)$$

which describes the stream function for a three-dimensional source in potential flow. In this expression, $\psi_0 = -Q/2\pi$, where Q is the strength of the source, i.e. the output.

For $\psi_0 = 0$, we obtain

$$\psi = -\frac{C}{k}\left(1 - \cos kz - k\int_0^z \sin k\sqrt{\zeta^2+r^2}\, ds\right) \qquad (10)$$

Thus, for the circumferential velocity v_ϕ, in conformity with (3), we have

$$v_\varphi = \frac{C}{r}\left(\cos kz + k\int_0^z \sin k\sqrt{\zeta^2+r^2}\, d\zeta\right) \qquad (11)$$

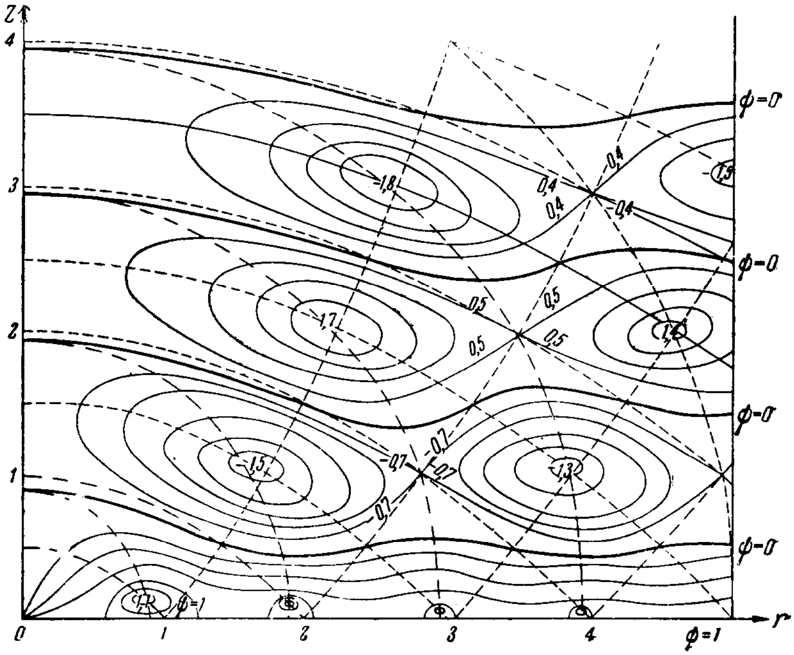
Substituting for ψ from (8) for this case, into equation (1), we obtain the equation for v_ϕ:

$$\frac{\partial^2 v_\varphi}{\partial z^2} + \frac{\partial^2 v_\varphi}{\partial r^2} + \frac{1}{r}\frac{\partial v_\varphi}{\partial r} + \left(k^2 - \frac{1}{r^2}\right) v_\varphi = 0 \tag{12}$$

We see that equation (11) describes a solution of this equation which would be characteristic of a potential flow ($k = 0$) in which

$$v_\varphi = \frac{C}{r} \qquad \left(C = \frac{\Gamma}{2\pi}\right)$$

This corresponds to an infinitesimally thin vortex line along the axis, where Γ is the circulation of the velocity along a closed curve embracing the axis z [1].



Note: O.F. Vasil'ev obtained another expression for ψ:

$$\psi = \psi_0 \left[1 + \frac{2k^2}{\pi}\int_0^\infty \frac{\sin \lambda z \, d\lambda}{\lambda(\lambda^2 - k^2)} - r\int_0^\infty \frac{\lambda \sin \lambda z}{\sqrt{k^2 - \lambda^2}} Y_1 (r\sqrt{k^2 - \lambda^2})\, d\lambda + \right.$$
$$\left. + kC\left[\frac{2}{\pi}\int_0^\infty \frac{\sin \lambda z \, d\lambda}{\lambda(\lambda^2 - k^2)} - r\int_0^\infty \frac{\sin \lambda z}{\lambda\sqrt{k^2 - \lambda^2}} Y_1 (r\sqrt{k^2 - \lambda^2})\, d\lambda\right]\right. \tag{13}$$

Comparing this expression with (9), and keeping in mind that

$$\int_0^\infty \frac{\sin \lambda z \, d\lambda}{\lambda(\lambda^2 - k^2)} = -\frac{\pi}{2k^2}(1 - \cos kz)$$

(the integral is considered as a Cauchy principal value), we derive the following equations

$$B(r, \lambda, k) = r\int_0^\infty \frac{\sin \lambda z}{\lambda\sqrt{k^2-\lambda^2}} Y_1(r\sqrt{k^2-\lambda^2})\,d\lambda = -\frac{1}{k}\int_0^z \sin k\sqrt{\zeta^2+r^2}\,d\zeta$$

$$C(r, z, k) = -\frac{\partial^2 B}{\partial z^2} = r\int_0^\infty \frac{\lambda \sin \lambda z}{\sqrt{k^2-\lambda^2}} Y_1(r\sqrt{k^2-\lambda^2})\,d\lambda = \frac{z\cos\sqrt{z^2+r^2}}{\sqrt{z^2+r^2}}$$

In these integrals it is necessary for $\lambda > k$ to replace

$$\frac{Y_1(r\sqrt{k^2-\lambda^2})}{\sqrt{k^2-\lambda^2}} \quad \text{with} \quad \frac{2}{\pi}\frac{K_1(r\sqrt{\lambda^2-k^2})}{\sqrt{\lambda^2-k^2}}$$

In the figure, the general form of the flow lines is shown, that is, the lines ψ = const, for $C = 0$, $k = \pi$. In constructing the curves, assistance was provided by M.M. Semchinov and N.V. Volzhanskii.

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